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OBJECTS WITH CLOSED DIAGONALS

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In a category with finite products, special types of subcategories $\mathscr{B}$ and classes $\mathscr{M}$ of monomorphisms are considered, such that a 'diagonal theorem' of the type ($X \in \mathscr{B} \Leftrightarrow \Delta_X \in \mathscr{M}$) holds.

Introduction

A space X is Hausdorff if and only if $\Delta_X = \{(x, x) \mid x \in X\}$ is closed in $X \times X$. General topologists provided analogues of this fact for other separation axioms (T_0, T_1, $T_{2\frac{1}{2}}$ for instance) by modifying the topology on $X \times X$ (see [16] for references). The general format for this procedure goes back to Salbany [15]: for any class $\mathscr{A}$ of spaces, a subset M of X is called *$\mathscr{A}$-closed* if M is the equalizer of two maps $f, g: X \to A$ with $A \in \mathscr{A}$; the modified topology to be considered is the coarsest topology in which $\mathscr{A}$-closed sets are closed.

For an epireflective subcategory $\mathscr{A}$ of topological spaces, Giuli and Hušek [6] succeeded to generalize all previous diagonal theorems by showing that the objects of the quotient-reflective hull of $\mathscr{A}$ are exactly those with an $\mathscr{A}$-closed diagonal. In this paper we shall show that this theorem can be shown even for an arbitrary category $\mathscr{C}$ with finite products (instead of topological spaces) and all its reflective subcategories which satisfy a certain weak exactness property (which is always satisfied for topological spaces); see Theorem 1.1 below. '$\mathscr{A}$-closed' has become '$\mathscr{A}$-regular' since the theorem has been freed of its topological context, and since '$\mathscr{A}$-regular' is the categorical 'regular' if $\mathscr{A}$ is the whole category. The 'quotient-reflective hull' has become the 'strongly epireflective hull' of $\mathscr{A}$, denoted, as in Universal Algebra, by $S(\mathscr{A})$.

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Under mild conditions on the category $\mathscr{C}$, $S(\mathscr{A})$ is indeed the least strongly epireflective subcategory of $\mathscr{C}$ containing $\mathscr{A}$; at the same time, with $\mathscr{B} := S(\mathscr{A})$ one has: (1) $\mathscr{A} \rightarrow \mathscr{B}$ preserves epimorphisms and (2) $\mathscr{A}$ is epireflective in $\mathscr{B}$ (even bireflective). There is a largest strongly epireflective subcategory with (1) and (2), called $D(\mathscr{A})$ and introduced by Hoffmann [9], and a largest one with (2), called $B(\mathscr{A})$ and introduced by Baron [1]. Naturally, both subcategories can be very useful when one wants to show cowellpoweredness of $\mathscr{A}$ (see [4, 6]). For $\mathscr{C} = \mathbf{Top}$ and $\mathscr{A}$ a closed-hereditary subcategory, Hoffmann [8] gave a diagonal characterization for the spaces in $D(\mathscr{A})$ which we shall show here for abstract categories; see Theorem 3.2.

Also for $B(\mathscr{A})$ a diagonal theorem is given (see Corollary 4.4) which, however is a direct consequence of a more general result which was obtained by Pumplün [13] a long time ago but published just recently [14]: for any 'good' class $\mathscr{M}$ (one which is of the form $\mathscr{E}^{\downarrow}$ in the sense of [5]), the authors of [14] characterize the objects X with $\Delta_X \in \mathscr{M}$. We pay special attention here to a particular class $\mathscr{M}$, the $\mathscr{A}$-strong monomorphism which, by a diagonal theorem, characterize the subcategory $E(\mathscr{A})$. One has the chain

$$\mathscr{A} \subseteq S(\mathscr{A}) \subseteq E(\mathscr{A}) \subseteq D(\mathscr{A}) \subseteq B(\mathscr{A}),$$

and all inclusions may be proper. The relationship between the various sorts of (mono)morphisms used in the diagonal theorems is summarized in the last section of the paper which also gives factorization theorems involving these morphisms.

We do not treat, in our abstract context, 'closedness' in terms of closure operators although the factorization theorems presented here pave the way for this procedure, but reserve the elaboration for a succeeding paper. We note, however, that Manes [11] defined 'perfect maps', 'compact objects', and 'Hausdorff objects' in concrete categories with a closure operator.

Finally we want to stress the point that for all the diagonal theorems presented in this paper only finite constructions are used. We do not have interesting examples in which this is essential. However, after finishing this work, Heath's [7] paper on the finite completeness of the category of locally-equiconnected spaces appeared; these are spaces X for which the inclusion of Δ_X into $X \times X$ is a cofibration. His proof uses widely general categorical methods, but not exclusively.

Throughout the paper, $\mathscr{C}$ is a category with finite products. For an object X,

$$\Delta_X : X \rightarrow X \times X$$

denotes the *diagonal of* X, and for a morphism $f : X \rightarrow Y$,

$$\Gamma_f : X \rightarrow X \times Y$$

is the *graph of* f (with $p\Gamma_f = 1_X$, $q\Gamma_f = f$ and projections p, q). Subcategories are always assumed to be full and replete (isomorphism-closed) and will be identified with their class of objects.

1. Objects with $\mathscr{A}$-regular diagonals

For a subcategory $\mathscr{A}$ of $\mathscr{C}$, a morphism $m: X \to Y$ is an *$\mathscr{A}$-regular monomorphism* if it is the equalizer of two morphisms $f, g: Y \to A$ with $A \in \mathscr{A}$. In order to characterize those objects X which have an $\mathscr{A}$-regular diagonal we consider the subcategory

$$S(\mathscr{A}) := \{X \in \mathscr{C} \mid \text{there is a monomorphism } X \to A \text{ with } A \in \mathscr{A}\};$$

if $\mathscr{A}$ is reflective with reflector r and reflexion $\rho: 1_{\mathscr{C}} \to r$, then

$$S(\mathscr{A}) = \{X \in \mathscr{C} \mid \rho_X \text{ is a monomorphism}\}.$$

If, moreover, $\mathscr{C}$ has (strong epi, mono)-factorizations, then $S(\mathscr{A})$ is the *strongly epireflective hull of $\mathscr{A}$ in $\mathscr{C}$*, and $\mathscr{A}$ is bireflective in $S(\mathscr{A})$.

For T and X in $\mathscr{C}$, we denote by

$$k_{T,X}: r(T \times X) \to rT \times rX$$

the *canonical morphism* with $uk_{T,X} = r(p)$, $vk_{T,X} = r(q)$ where p, q are projections of $T \times X$ and u, v projections of $rT \times rX$.

Theorem 1.1. *For a reflective subcategory $\mathscr{A}$ of $\mathscr{C}$ and an object X in $\mathscr{C}$, let the canonical morphism $k_{T,X}$ be a monomorphism for all T in $\mathscr{C}$. Then X belongs to $S(\mathscr{A})$ if and only if Δ_X is $\mathscr{A}$-regular.*

Proof. For $X \in S(\mathscr{A})$, Δ_X is the equalizer of $\rho_X s$ and $\rho_X t$ (where s, t are projections of $X \times X$) since ρ_X is a monomorphism. Vice versa, if Δ_X is the equalizer of two morphisms $f, g: X \times X \to A$ with $A \in \mathscr{A}$ we must show that ρ_X is a monomorphism. So let $\rho_X x = \rho_X y$ for morphisms $x, y: T \to X$.

First, one easily checks that

$$\begin{array}{ccc} T & \xrightarrow{x} & X \\ {\scriptstyle \Gamma_x}\downarrow & & \downarrow{\scriptstyle \Delta_X} \\ T \times X & \xrightarrow{x \times 1} & X \times X \end{array} \tag{1}$$

commutes. Secondly, with $k = k_{T,X}$ and $\rho = \rho_{T \times X}$, from

$$uk\rho\Gamma_x = r(p)\rho\Gamma_x = \rho_T p\Gamma_x = \rho_T = uk\rho\Gamma_y ,$$
$$vk\rho\Gamma_x = \rho_X q\Gamma_x = \rho_X x = \rho_X y = vk\rho\Gamma_y ,$$

one can derive that

$$\rho\Gamma_x = \rho\Gamma_y \tag{2}$$

since k is a monomorphism. Thirdly, we claim that

$$f(x \times 1)\Gamma_y = g(x \times 1)\Gamma_y . \tag{3}$$

For that we consider the morphisms f_x, g_x which make the diagram

$$\begin{tikzcd} T \times X \arrow[r, "x \times 1"] \arrow[d, "\rho"'] & X \times X \arrow[d, shift right, "f"'] \arrow[d, shift left, "g"] \\ r(T \times X) \arrow[r, shift left, "f_x"] \arrow[r, shift right, "g_x"'] & A \end{tikzcd}$$

commute in the obvious sense. Then, with (1) and (2), we obtain (3):

$$\begin{aligned} f(x \times 1)\Gamma_y &= f_x\rho\Gamma_y = f_x\rho\Gamma_x = f(x \times 1)\Gamma_x \\ &= f\Delta_X x = g\Delta_X x = g(x \times 1)\Gamma_x \\ &= g_x\rho\Gamma_x = g_x\rho\Gamma_y = g(x \times 1)\Gamma_y . \end{aligned}$$

Finally, by the equalizer property, there is a morphism z rendering

$$\begin{tikzcd} T \arrow[r, "z"] \arrow[d, "\Gamma_y"'] & X \arrow[d, "\Delta_X"] \\ T \times X \arrow[r, "x \times 1"] & X \times X \end{tikzcd}$$

commutative. But this gives

$$z = s\Delta_X z = s(x \times 1)\Gamma_y = xp\Gamma_y = x ,$$
$$z = t\Delta_X z = t(x \times 1)\Gamma_y = q\Gamma_y = y ,$$

so $x = y$. $\square$

In the above proof, it suffices to take T to be a generator of the category $\mathscr{C}$. If, at the same time, T is a terminal object in $\mathscr{C}$, then the projection $q : T \times X \to X$ is an isomorphism, hence also $vk_{T,X} = r(q)$ is one, so $k_{T,X}$ is monic. So from Theorem 1.1 we get

Corollary 1.2. *Let the terminal object in $\mathscr{C}$ be a generator of $\mathscr{C}$. If $\mathscr{A}$ is reflective in $\mathscr{C}$, then $S(\mathscr{A})$ contains exactly the objects with $\mathscr{A}$-regular diagonals.* □

Remarks. (1) The terminal object T of $\mathscr{C}$ is certainly a generator of $\mathscr{C}$ if there is a faithful functor $U : \mathscr{C} \to \mathbf{Set}$ such that $UT \neq \emptyset$ and every constant mapping $UT \to UX$ lifts to a $\mathscr{C}$-morphism $T \to X$. Specifically, $\mathscr{C} = \mathbf{Top}$ has this property. Therefore Corollary 1.2 generalizes [6, Theorem 2.2] where the corollary was proved for epireflective subcategories of **Top**. However, Theorem 1.1 is not confined to applications in topology. For instance, for $\mathscr{C} = \mathbf{Grp}$ and $\mathscr{A} = \mathbf{Ab}$, $k_{T,X}$ is a monomorphism for all T and X. Therefore, a group X is abelian if and only if $\Delta_X = \{(x, y) \mid f(x, y) = g(x, y)\}$ for two homomorphisms $f, g : X \times X \to A$ into an abelian group A.

(2) The condition that $k_{T,X}$ be a monomorphism for all T and X is not a necessary condition for the equality $S(\mathscr{A}) = \{X \mid \Delta_X \text{ is } \mathscr{A}\text{-regular}\}$. For example, in $\mathscr{C} = \mathbf{Set}^{\mathrm{op}} \times \mathbf{Set}^{\mathrm{op}}$, the latter equation holds for $\mathscr{A} = S(\mathscr{A}) = \{(U, V) \mid U \neq \emptyset \neq V$ or $U = \emptyset = V\}$ although $k_{T,X}$ is not always a monomorphism in $\mathscr{C}$ (consider, for instance, $T = (U, \emptyset)$ and $X = (\emptyset, V)$ with $U \neq \emptyset \neq V$).

(3) We did not find any example of a reflective subcategory $\mathscr{A}$ of a category with finite products for which $S(\mathscr{A}) = \{X \mid \Delta_X \text{ is } \mathscr{A}\text{-regular}\}$ fails.

(4) Calling a split monomorphism $d : X \to A$ with $A \in \mathscr{A}$ an *$\mathscr{A}$-split monomorphism* we have that $\mathscr{A}$-split monomorphisms are $\mathscr{A}$-regular. With this notation one has the following (trivial) diagonal characterization *for every reflective subcategory $\mathscr{A}$ of $\mathscr{C}$: $X \in \mathscr{A}$ holds if and only if Δ_X is an $\mathscr{A}$-split monomorphism* since the latter condition just means $X \times X \in \mathscr{A}$.

2. Objects with $\mathscr{A}$-strong diagonals

Let $\mathscr{A}$ be a subcategory of $\mathscr{C}$. A morphism $p : U \to V$ in $\mathscr{C}$ is called *$\mathscr{A}$-cancellable* if $fp = gp$ with $f, g : V \to A$, $A \in \mathscr{A}$, implies $f = g$. We do not call these morphisms *$\mathscr{A}$-epimorphisms* as in [9, 14], but reserve this name, as many other authors do, for those $\mathscr{A}$-cancellable morphisms $p : U \to V$ with $V \in \mathscr{A}$. The class $\mathrm{Can}_{\mathscr{C}}(\mathscr{A})$ of all $\mathscr{A}$-cancellable morphisms in $\mathscr{C}$ contains all epimorphisms of $\mathscr{C}$, is closed under composition and colimits, and is stable under (multiple) pushouts. We notice that $\mathrm{Can}_{\mathscr{C}}(\mathscr{A}) = \mathrm{Can}_{\mathscr{C}}(S(\mathscr{A}))$.

Let $\mathscr{E}$ be a subclass of Mor $\mathscr{C}$. An object X in $\mathscr{C}$ is called *$\mathscr{E}$-Hausdorff* (cf. [13, 14]) if every $p \in \mathscr{E}$ is $\{X\}$-cancellable. The class $\mathrm{Haus}_{\mathscr{C}}(\mathscr{E})$ of all $\mathscr{E}$-Hausdorff objects in $\mathscr{C}$ is closed under mono-sources in $\mathscr{C}$, hence strongly epireflective if sources in $\mathscr{C}$ have (strong epi, mono-source)-factorizations. For every $\mathscr{A}$ and $\mathscr{E}$ one has

$$\mathscr{E} \subseteq \mathrm{Can}_{\mathscr{C}}(\mathscr{A}) \Leftrightarrow \mathscr{A} \subseteq \mathrm{Haus}_{\mathscr{C}}(\mathscr{E}),$$

so $\mathrm{Can}_{\mathscr{C}}(-)$ and $\mathrm{Haus}_{\mathscr{C}}(-)$ define a Galois correspondence. The objects of those classes $\mathscr{A}$ which are closed under this correspondence can be characterized by a diagonal theorem. For this denote by $\mathscr{E}_{\perp}$ $(=\mathscr{E}^{\downarrow}$ in [5]) the class of all morphisms m with $p \perp m$ for all $p \in \mathscr{E}$, that is: whenever $mg = hp$ with $p \in \mathscr{E}$, there is a unique t with $tp = g$ and $mt = h$.

Proposition 2.1 (cf. Pumplün [13], Pumplün and Röhrl [14]). *For every subclass $\mathscr{E} \subseteq \mathrm{Mor}\,\mathscr{C}$ and every object X in $\mathscr{C}$, the following assertions are equivalent*:

(i) $X \in \mathrm{Haus}_{\mathscr{C}}(\mathscr{E})$;

(ii) $\Delta_X \in \mathscr{E}_{\perp}$;

(iii) $\Gamma_f \in \mathscr{E}_{\perp}$ *for all* $f : Z \to X$;

(iv) *If m is an equalizer of a pair $f, g : Z \to X$, then $m \in \mathscr{E}_{\perp}$.*

Proof. Straight exercise along the implications (i)$\Rightarrow$(iv)$\Rightarrow$(iii)$\Rightarrow$(ii)$\Rightarrow$(i). □

Since every epimorphism in $\mathscr{C}$ is $\mathscr{A}$-cancellable, every monomorphism in $(\mathrm{Can}_{\mathscr{C}}(\mathscr{A}))_{\perp}$ is a strong monomorphism in $\mathscr{C}$ and will therefore be called *$\mathscr{A}$-strong*. Every $\mathscr{A}$-regular monomorphism is $\mathscr{A}$-strong. (The converse fails even when $\mathscr{A} = \mathscr{C}$; see [10].) For

$$E(\mathscr{A}) := \mathrm{Haus}_{\mathscr{C}}(\mathrm{Can}_{\mathscr{C}}(\mathscr{A}))$$

we have $S(\mathscr{A}) \subseteq E(\mathscr{A})$ (since $\mathrm{Can}_{\mathscr{C}}(\mathscr{A}) = \mathrm{Can}_{\mathscr{C}}(S(\mathscr{A}))$), and Proposition 2.1, (i)$\Leftrightarrow$(ii), gives

Corollary 2.2. *For every subcategory $\mathscr{A}$ of $\mathscr{C}$ and every object X in $\mathscr{C}$, X belongs to $E(\mathscr{A})$ if and only if Δ_X is $\mathscr{A}$-strong.* □

Example. Let $\mathscr{C} = \mathbf{Top}$ and $\mathscr{A} = S(\mathscr{D})$ where $\mathscr{D}$ consists of all powers of the two-point discrete space. Then the Tychonov corkscrew [17, #90] belongs to $E(\mathscr{A})$ but not to $S(\mathscr{A}) = \mathscr{A}$ (see [3]). Consequently, its diagonal is an $\mathscr{A}$-strong (split-)monomomphism by Corollary 2.2, but not $\mathscr{A}$-regular by Theorem 1.1 (note that the condition on the canonical morphisms is satisfied here according to Remark (1) after Corollary 1.2). For $\mathscr{C} = \mathbf{Top}$ and $\mathscr{A}$ the category of Urysohn spaces, one can find an example of an $\mathscr{A}$-strong but not $\mathscr{A}$-regular monomorphism in [2] (this example goes back to [16]).

Remark. One may have $E(\mathscr{A}) = S(\mathscr{A})$, but still the existence of $\mathscr{A}$-strong monomorphisms which are not $\mathscr{A}$-regular: consider any category $\mathscr{C}$ in which there are strong but non-regular monomorphisms, and take $\mathscr{A} = \mathscr{C}$.

We conclude this section with a sufficient criterion for $E(\mathscr{A}) = \mathscr{C}$:

Proposition 2.3. *Suppose there is a bireflective subcategory $\mathscr{B}$ of $\mathscr{C}$ such that $\mathscr{A} \subseteq \mathscr{B} \subseteq S(\mathscr{A}) \subseteq \mathscr{C}$. Then $E(\mathscr{A}) = \mathscr{C}$.*

Proof. It suffices to show that every $\mathscr{A}$-cancellable morphism $p : U \to V$ in $\mathscr{C}$ is an epimorphism in $\mathscr{C}$. So, for $f, g : V \to X$, let $fp = gp$. Then $m\rho_X fp = m\rho_X gp$ with the (monic) reflexion ρ_X into $\mathscr{B}$ and a monomorphism m with codomain in $\mathscr{A}$. Now $m\rho_X f = m\rho_X g$, hence $f = g$ follows. □

Of course, Proposition 2.3 can be slightly generalized since the monomorphism m in the above proof may be replaced by an arbitrary mono-source with base in $\mathscr{A}$. In any case, for $\mathscr{C} = \mathbf{Top}$, the sufficient condition of Proposition 2.3 is also necessary:

Corollary 2.4. *For every subcategory $\mathscr{A}$ of* **Top** *one has $E(\mathscr{A}) = \mathbf{Top}$ if and only if the epireflective hull of $\mathscr{A}$ is bireflective in* **Top.**

Proof. For 'if' apply Proposition 2.3, replacing $\mathscr{A}$ by the subcategory of all products of spaces in $\mathscr{A}$. To show 'only if', suppose the epireflective hull $\bar{S}(\mathscr{A})$ (see Corollary 4.4 below) is not bireflective in **Top**. Then $\mathscr{A}$ can only contain T_0-spaces, so $E(\mathscr{A}) \subseteq E(T_0 - \mathbf{Top}) = T_0 - \mathbf{Top} \subsetneq \mathbf{Top}$. □

3. Objects with $\mathscr{A}$-straight diagonals

For a subcategory $\mathscr{A}$ of $\mathscr{C}$, let $\mathrm{Epi}_{\mathscr{C}}(\mathscr{A})$ be the class of all $\mathscr{A}$-epimorphisms in $\mathscr{C}$ and define

$$D(\mathscr{A}) := \mathrm{Haus}_{\mathscr{C}}(\mathrm{Epi}_{\mathscr{C}}(\mathscr{A})) \quad \text{(see Section 2);}$$

one has $\mathscr{A} \subseteq S(\mathscr{A}) \subseteq E(\mathscr{A}) \subseteq D(\mathscr{A}) \subseteq \mathscr{C}$. If $\mathscr{B}$ is one of these three intermediate categories, then

(1) $\mathscr{A} \to \mathscr{B}$ preserves epimorphisms,

(2) $\mathscr{A}$ is epireflective in $\mathscr{B}$ if $\mathscr{A}$ is reflective in $\mathscr{C}$,

(3) $\mathscr{B}$ is strongly epireflective in $\mathscr{C}$ if every source in $\mathscr{C}$ factors through a strong epimorphism and a mono-source.

In addition:

Proposition 3.1 (cf. Hoffmann [9]). *For any subcategory $\mathscr{B}$ of $\mathscr{C}$ with $\mathscr{A} \subseteq \mathscr{B}$, such that* (1) *$\mathscr{A} \to \mathscr{B}$ preserves epimorphisms,* (2) *$\mathscr{A}$ is epireflective in $\mathscr{B}$, and* (3) *$\mathscr{B}$ is strongly epireflective in $\mathscr{C}$, one has $S(\mathscr{A}) \subseteq \mathscr{B} \subseteq D(\mathscr{A})$.*

Proof. For $X \in S(\mathscr{A})$, the reflexion $\sigma_X : X \to sX$ into $\mathscr{B}$ is a strong epimorphism in $\mathscr{A}$, but also a monomorphism since there is a monomorphism $X \to A$, $A \in \mathscr{A}$, through which σ_X factors; so $X \in \mathscr{B}$. For $B \in \mathscr{B}$, let $p: U \to A$ be an $\mathscr{A}$-epimorphism, and let f, g satisfy $fp = gp$. Then p factors in the form $p = e\rho_{sU}\sigma_U$ where ρ_{sU} is a reflexion into $\mathscr{A}$. It follows that $fe\rho_{sU} = ge\rho_{sU}$ since σ_U is a reflexion. Then $fe = ge$ since ρ_{sU} is an epimorphism in $\mathscr{B}$. Since p is $\mathscr{A}$-epimorphic, e is an epimorphism in $\mathscr{A}$ and by (1), also in $\mathscr{B}$. So $f = g$ follows. $\square$

Every monomorphism in $(\mathrm{Epi}_{\mathscr{C}}(\mathscr{A}))_{\perp}$ is a strong monomorphism and will be called *$\mathscr{A}$-straight*. $\mathscr{A}$-strong monomorphisms are $\mathscr{A}$-straight. From Proposition 2.1 we derive

Theorem 3.2. *For every subcategory $\mathscr{A}$ of $\mathscr{C}$ and every object $X \in \mathscr{C}$, the following assertions are equivalent*:

(i) $X \in D(\mathscr{A})$;

(ii) Δ_X *is $\mathscr{A}$-straight*;

(iii) $h^{-1}\Delta_X$ *is $\mathscr{A}$-straight for all $h: Z \to X \times X$ such that the pullback exists.*

For $\mathscr{C}$ finitely complete and $\mathscr{A}$ reflective, also (iv) *and* (v) *are equivalent to* (i)–(iii):

(iv) $h^{-1}\Delta_X$ *is $\mathscr{A}$-straight for all $h: A \to X \times X$ with $A \in \mathscr{A}$*;

(v) $h^{-1}\Delta_X$ *is $\mathscr{A}$-strong for all $h: A \to X \times X$ with $A \in \mathscr{A}$.*

Proof. The equivalence of (i)–(iii) follows from the equivalence of statements (i), (ii), (iv) in Proposition 2.1; one just needs to convince himself that, for every $h: Z \to X \times X$, the pullback $h^{-1}\Delta_X$ exists if and only if the equalizer of ph and qh (p, q projections) exists, and that then both constructions coincide.

(ii)$\Rightarrow$(v). Since Δ_X is $\mathscr{A}$-straight, also $d = h^{-1}\Delta_X$ is (with $h: A \to X \times X$). In order to show that d is even $\mathscr{A}$-strong, assume $ge = df$ with an $\mathscr{A}$-cancellable morphism $e: U \to V$. Since $\mathscr{A}$ is reflective and $A \in \mathscr{A}$, g factorizes in the form $g = \bar{g}\rho_V$, and $\rho_V e$ is an $\mathscr{A}$-epimorphism. Hence there is a morphism s with $ds = \bar{g}$. So, with $t = s\rho_V$, one has $dt = g$ and $tp = f$ since d is monic.

(v)$\Rightarrow$(iv). Trivial.

(iv)$\Rightarrow$(ii). If $\Delta_X g = hp$ with an $\mathscr{A}$-epimorphism $p: U \to A$, then p factorizes through $d = h^{-1}\Delta_X$. Therefore d is $\mathscr{A}$-epimorphic, but by assumption also $\mathscr{A}$-straight, hence an isomorphism. Now, with h' the pullback of h along Δ_X and with $t = h'd^{-1}$, one has $\Delta_X t = h$ and $tp = g$. So Δ_X is $\mathscr{A}$-straight. $\square$

Remark. The equivalence (i)$\Leftrightarrow$(v) of Theorem 3.2 was obtained before by Hoffmann [8] in case $\mathscr{C} = \mathbf{Top}$ and $\mathscr{A} = \mathbf{CompHaus}$. In that case $D(\mathscr{A})$ is the category of t_2'-spaces as introduced in [9] (t_2' coincides with McCord's [12] 'weak Hausdorff' axiom t_2 for k-spaces). Since there are non-Hausdorff t_2'-spaces whereas all spaces in $E(\mathscr{A})$ are Hausdorff, one has $E(\mathscr{A}) \subsetneq D(\mathscr{A})$. Consequently, there are $\mathscr{A}$-straight (split-)monomorphisms which are not $\mathscr{A}$-strong.

4. Baron's maximal intermediate category

Let $\mathscr{A}$ be a reflective subcategory of $\mathscr{C}$ such that there is an intermediate category $\mathscr{B}_0$ with $\mathscr{A}$ epireflective in $\mathscr{B}_0$ and $\mathscr{B}_0$ epireflective in $\mathscr{C}$. (If morphisms in $\mathscr{C}$ factor (strong epi, mono), then $S(\mathscr{A})$ may serve as $\mathscr{B}_0$.) Using Baron's [1] idea one defines

$$B(\mathscr{A}) := \mathrm{Haus}_{\mathscr{C}}\ (\{\rho_Y \mid Y \in \mathscr{B}_0\})$$

where ρ is the reflexion into $\mathscr{A}$. Trivially $D(\mathscr{A}) \subseteq B(\mathscr{A})$; also $\mathscr{B}_0 \subseteq B(\mathscr{A})$ since $\mathscr{A}$ is epireflective in $\mathscr{B}_0$; and $B(\mathscr{A})$ is (strongly) epireflective in $\mathscr{C}$ whenever sources in $\mathscr{C}$ factor through a (strong) epimorphism and a mono-source.

The following proposition shows that $B(\mathscr{A})$ is independent from the choice of $\mathscr{B}_0$ as soon as $B(\mathscr{A})$ is reflective in $\mathscr{C}$:

Proposition 4.1. *Let the reflective subcategory $\mathscr{A}$ of $\mathscr{C}$ admit an intermediate category $\mathscr{B}_0$ as above. Then $\mathscr{A}$ is epireflective in $B(\mathscr{A})$, and for any reflective subcategory $\mathscr{B}$ of $\mathscr{C}$ which contains $\mathscr{A}$ as an epireflective subcategory one has $\mathscr{B} \subseteq B(\mathscr{A})$.*

Proof. For $Y \in B(\mathscr{A})$ we first have to show that ρ_Y is an epimorphism in $B(\mathscr{A})$. But, up to an isomorphism, ρ_Y decomposes as $\rho_Y = \rho_{sY}\sigma_Y$ where σ_Y is the epireflexion into $\mathscr{B}_0$. If $f\rho_Y = g\rho_Y$ with $f, g: rY \to X$, $X \in B(\mathscr{A})$, then $f\rho_{sY} = g\rho_{sY}$. Therefore $f = g$ by definition of $B(\mathscr{A})$.

For $\mathscr{B}$ as given in Proposition 4.1 let now $Y \in \mathscr{B}_0$, let σ_Y be the reflexion into $\mathscr{B}$, and assume $f\rho_Y = g\rho_Y$ as above, but with $X \in \mathscr{B}$. Since σ_Y is $\mathscr{B}$-epimorphic one has $f\rho_{sY} = g\rho_{sY}$. But ρ_{sY} is an epimorphism in $\mathscr{B}$, so $f = g$. This shows $X \in B(\mathscr{A})$. □

Corollary 4.2. *$\mathscr{A}$ is epireflective in $\mathscr{C}$ if and only if $B(\mathscr{A}) = \mathscr{C}$.* □

Corollary 4.3 (cf. Hoffmann [9]). *$D(\mathscr{A}) = B(\mathscr{A})$ holds if and only if $\mathscr{A} \to B(\mathscr{A})$ preserves epimorphisms.*

Proof. Note that assumption (3) of Proposition 3.1 is not needed in order to derive $\mathscr{B} \subseteq D(\mathscr{A})$. □

If every morphism in $\mathscr{C}$ factors (epi, strong mono), then the reflective subcategory $\mathscr{A}$ has an *epireflective hull*, namely

$$\bar{S}(\mathscr{A}) = \{X \mid \rho_X \text{ is a strong monomorphism}\}\,,$$

and $\mathscr{A}$ is bireflective in $\bar{S}(\mathscr{A})$. So $\bar{S}(\mathscr{A})$ is the minimal choice for $\mathscr{B}_0$ above. There is also a maximal choice, namely $B(\mathscr{A})$, if it is epireflective in $\mathscr{C}$. From Propositions 2.1 and 4.1 we therefore obtain

Corollary 4.4. *If $\bar{S}(\mathscr{A})$ and $B(\mathscr{A})$ are admissible choices for $\mathscr{B}_0$, then*

$$\begin{aligned} X \in B(\mathscr{A}) \;&\Leftrightarrow\; \Delta_X \in \{\rho_Y \mid Y \in \bar{S}(\mathscr{A})\}_\perp \\ &\Leftrightarrow\; \Delta_X \in \{\rho_Y \mid Y \in B(\mathscr{A})\}_\perp \end{aligned}$$

where ρ is the reflexion into $\mathscr{A}$. □

Remarks. (1) By Corollaries 4.2 and 4.3 it is trivial that, in general, $D(\mathscr{A}) \subsetneqq B(\mathscr{A})$: consider $\mathscr{C} = \mathbf{Top}$ and $\mathscr{A} = \mathbf{Haus}$. Consequently, there are split-monomorphisms in $\{\rho_Y \mid Y \in B(\mathscr{A})\}_\perp$ ($\subseteq \{\rho_Y \mid Y \in \bar{S}(\mathscr{A})\}_\perp$) which are not $\mathscr{A}$-straight.

(2) We do not have a non-trivial diagonal characterization for $\bar{S}(\mathscr{A})$. Since $\mathscr{A} \subseteq \bar{S}(\mathscr{A}) \subseteq S(\mathscr{A})$ one would have to find a suitable subclass $\mathscr{M}$ of $\mathscr{A}$-regular monomorphisms which contains all $\mathscr{A}$-split monomorphisms (see Section 1).

5. Factorizations

In the following, let $\mathscr{A}$ be always closed under finite products. We denote by $\mathrm{Split}_{\mathscr{C}}(\mathscr{A})$ ($\mathrm{Reg}_{\mathscr{C}}(\mathscr{A})$, $\mathrm{Strong}_{\mathscr{C}}(\mathscr{A})$) the class of $\mathscr{A}$-split ($\mathscr{A}$-regular, $\mathscr{A}$-strong resp.) monomorphisms in $\mathscr{C}$; one has

$$\mathrm{Split}_{\mathscr{C}}(\mathscr{A}) \subseteq \mathrm{Reg}_{\mathscr{C}}(\mathscr{A}) \subseteq \mathrm{Strong}_{\mathscr{C}}(\mathscr{A}) \tag{$*$}$$

for every subcategory $\mathscr{A}$ of $\mathscr{C}$. For any subclass $\mathscr{M}$ of morphisms in $\mathscr{C}$, let $\mathscr{M}^\perp$ ($=\mathscr{M}^\uparrow$ in [5]) be the class of all morphisms p in $\mathscr{C}$ with $p \perp m$ for all $m \in \mathscr{M}$ (see Section 2).

Lemma 5.1. $(\mathrm{Split}_{\mathscr{C}}(\mathscr{A}))^\perp = (\mathrm{Reg}_{\mathscr{C}}(\mathscr{A}))^\perp = (\mathrm{Strong}_{\mathscr{C}}(\mathscr{A}))^\perp = \mathrm{Can}_{\mathscr{C}}(\mathscr{A})$.

Proof. We just need to show that $(\mathrm{Split}_{\mathscr{C}}(\mathscr{A}))^\perp \subseteq \mathrm{Can}_{\mathscr{C}}(\mathscr{A})$ since then

$$\begin{aligned} (\mathrm{Split}_{\mathscr{C}}(\mathscr{A}))^\perp &\subseteq \mathrm{Can}_{\mathscr{C}}(\mathscr{A}) \subseteq (\mathrm{Strong}_{\mathscr{C}}(\mathscr{A}))^\perp \\ &\subseteq (\mathrm{Reg}_{\mathscr{C}}(\mathscr{A}))^\perp \subseteq (\mathrm{Split}_{\mathscr{C}}(\mathscr{A}))^\perp \end{aligned}$$

will follow formally. If $p: U \to V$ belongs to $(\mathrm{Split}_{\mathscr{C}}(\mathscr{A}))^\perp$ and if $fp = gp$ for $f, g: V \to A$, $A \in \mathscr{A}$, then $\Delta_A k = hp$ with $k = fp$ and h the induced morphism $V \to A \times A$ with components f, g. Since $A \times A \in \mathscr{A}$ one has a 'diagonal' t with $\Delta_A t = h$, so $t = f = g$ follows. □

For $\mathcal{A} = \mathcal{C}$, the above lemma is essentially in [5]. W. Lawvere mentioned the following consequence to us:

Proposition 5.2. *Let $\mathcal{M}$ be one of the three classes in* (*), *and let $\mathcal{E}$ be a class such that every morphism factors $(\mathcal{E}, \mathcal{M})$ and that $e \perp m$ holds for all $e \in \mathcal{E}$ and $m \in \mathcal{M}$. Then $\mathcal{E} = \mathrm{Can}_{\mathcal{C}}(\mathcal{A})$.*

Proof. From the assumption, one has $\mathcal{E} = \mathcal{M}^{\perp}$, so Lemma 5.1 applies. (It must be assumed here that $\mathcal{E}$ is closed under composition with isomorphisms.) □

$\mathcal{C}$ is *$\mathcal{M}$-complete* if pullbacks of morphisms in $\mathcal{M}$ along arbitrary morphisms and multiple pullbacks of arbitrarily large families of morphisms in $\mathcal{M}$ exist and belong to $\mathcal{M}$. This is equivalent to the condition that one has locally orthogonal factorizations of sinks through $\mathcal{M}$-morphisms (see [18] in the dual situation). Since $\mathrm{Strong}_{\mathcal{C}}(\mathcal{A})$ is closed under (multiple) pullbacks and contained in the class of all strong monomorphisms of $\mathcal{C}$ we conclude with Lemma 5.1:

Theorem 5.3. *Let every sink in $\mathcal{C}$ factor into an epi-sink and a strong monomorphism. Then every sink in $\mathcal{C}$ factors into an $\mathcal{A}$-cancellable sink and an $\mathcal{A}$-strong monomorphism.* □

(A sink $(p_i : U_i \to V)_{i \in I}$ is an epi-sink ($\mathcal{A}$-cancellable sink) if $fp_i = gp_i$ for all $i \in I$ with $f, g : V \to W$ (and $W \in \mathcal{A}$) implies $f = g$.)

Corollary 5.4. *Let $\mathcal{C}$ be complete and wellpowered with respect to strong monomorphisms. Then every morphism in $\mathcal{C}$ has a $(\mathrm{Can}_{\mathcal{C}}(\mathcal{A}), \mathrm{Strong}_{\mathcal{C}}(\mathcal{A}))$-factorization.* □

Remark. Under the conditions of the corollary, with $\mathcal{M}$ the class of all $\mathcal{A}$-straight monomorphisms, one also has $(\mathcal{M}^{\perp}, \mathcal{M})$-factorizations of $\mathcal{C}$-morphisms. However, we do not have a good characterization of the morphisms in $\mathcal{M}^{\perp}$ (which contains the class of $\mathcal{A}$-epimorphisms).

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